MATH 4060 MIDTERM EXAM (FALL 2016)

Name: ____ Student ID: ___________________________

Answer all questions. Write your answers on this question paper. No books, notes or calculators are allowed. Time allowed: 105 minutes.

1. Weierstrass's theorem states that a continuous function on $[-1, 1]$ can be uniformly approximated by polynomials there. Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} \colon |z| \leq 1\}$ be the closed unit disc centered at the origin. Can every continuous function on $\overline{\mathbb{D}}$ be approximated uniformly on $\overline{\mathbb{D}}$ by polynomials in the complex variable z? Explain your answer. (10 points)

2. (a) Let

$$f(x) = \frac{1}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

Using contour integrals, show that its Fourier transform is

$$\widehat{f}(\xi) = \pi e^{-2\pi|\xi|},$$

where $\widehat{f}$ is defined by the formula $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx$ for all $\xi \in \mathbb{R}$. (16 points)

(b) (i) State (without proof) Poisson summation formula. You should state clearly a set of assumptions under which the conclusion of the theorem holds.

(ii) Using the version of Poisson summation formula you stated in part (i), evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

(14 points)

3. Suppose $k \in \mathbb{N}$. Let E_k be the canonical factor, defined by

$$E_k(z) = (1 - z) \exp\left(\sum_{j=1}^{k} \frac{z^j}{j}\right) \qquad \text{for } z \in \mathbb{C}.$$

(a) Show that

$$|E_k(z)| \geq e^{-2|z|^{k+1}} \qquad \text{for all } |z| \leq \frac{1}{2}.$$

(12 points)

(b) Show that if $z \in \mathbb{C}$, and $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers satisfying both conditions below:

- $|a_n| \geq 2|z|$ for all $n \in \mathbb{N}$,
- $\sigma := \displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^k} < \infty$,

then

$$\left| \prod_{n=1}^{\infty} E_k \left(\frac{z}{a_n} \right) \right| \geq e^{-\sigma|z|^k}.$$

(You do not need to prove the convergence of the infinite product on the left hand side.) (8 points)

4. For each of the following statements, determine whether it is true or false. If it is true, give a proof; if it is false, show that it is false.

   (a) The infinite product
$$\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)$$
   converges for all $z \in \mathbb{C}$, and defines an entire function that vanishes to order 1 at all the positive integers. (10 points)

(b) If P is a monic polynomial (i.e. a polynomial of the form $z^n + c_{n-1}z^{n-1} + \cdots + c_0$ for some $n \in \mathbb{N} \cup \{0\}$ and some coefficients $c_0, c_1, \ldots, c_{n-1} \in \mathbb{C}$) and $P(z) \neq 0$ whenever $|z| \geq 1$, then

$$\int_0^{2\pi} \log |P(e^{it})| dt = 0.$$

(14 points)

5. Suppose f is entire, not identically zero, and each zero of f occurs with an even multiplicity. Show that there exists an entire function g such that $g^2 = f$. (16 points)

End of paper